

# On dimensionally exotic maps

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## Abstract

We call a value  $y = f(x)$  of a map  $f : X \rightarrow Y$  dimensionally regular if  $\dim X \leq \dim(Y \times f^{-1}(y))$ . It was shown in [5] that if a map  $f : X \rightarrow Y$  between compact metric spaces does not have dimensionally regular values, then  $X$  is a Boltyanskii compactum, i.e. a compactum satisfying the equality  $\dim(X \times X) = 2 \dim X - 1$ . In this paper we prove that every Boltyanskii compactum  $X$  of dimension  $\dim X \geq 6$  admits a map  $f : X \rightarrow Y$  without dimensionally regular values. Also we exhibit a 4-dimensional Boltyanskii compactum for which every map has a dimensionally regular value.

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## 1 Introduction

Throughout this paper we assume that maps are continuous and spaces are separable metrizable. We recall that a *compactum* means a compact metric space. By dimension  $\dim X$  of a space  $X$  we assume the covering dimension.

The famous Sard theorem states that a smooth map of closed manifolds  $f : X \rightarrow Y$  has a regular value. How far Sard's theorem can be extended for mappings of compact metric spaces? We study this question from dimension theoretic point of view. We call a value  $y \in Y$  for a continuous mapping of compact metric spaces  $f : X \rightarrow Y$  to be *dimensionally regular* if

$$\dim X \leq \dim(Y \times f^{-1}(y)).$$

Note that a regular value of a smooth map is dimensionally regular. The question we study is whether every continuous map between compacta has a dimensionally regular value.

In our previous paper [5] we proved that this question has an affirmative answer if compactum  $X$  has the property  $\dim(X \times X) = 2 \dim X$ . It is known that not all compact

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metric spaces are such. The first example of a compactum which do not satisfies this equality was constructed by Boltyanskii [1]. His example was 2-dimensional with the dimension of the square equal 3. We call a compactum  $X$  a *Boltyanskii compactum* if it has the property  $\dim(X \times X) \neq 2 \dim X$ . It is known that for Boltyanskii compacta necessarily  $\dim(X \times X) = 2 \dim X - 1$ .

One of the main discovery of [5] is the existence of continuous maps  $f : X \rightarrow Y$  without dimensionally regular values. We called such maps *dimensionally exotic*. We proved the following

**Theorem 1.1** ([5]) *For every  $n \geq 4$  there is an  $n$ -dimensional Boltyanskii compactum  $X$  admitting a dimensionally exotic map  $f : X \rightarrow Y$  to a 2-dimensional compactum  $Y$ .*

The question remained unanswered whether every Boltyanskii compactum admits an exotic map. We address it in this paper. Namely, we prove the following:

**Theorem 1.2** *Every finite dimensional Boltyanskii compactum  $X$  with  $\dim X \geq 6$  admits a dimensionally exotic map  $f : X \rightarrow Y$  to a 4-dimensional compactum  $Y$ .*

In [5] we observed that every continuous map of a compactum of dimension  $\leq 3$  has dimensionally regular values. It turns out that in the case of dimension 4 there is no uniform answer.

**Theorem 1.3** *There is a 4-dimensional Boltyanskii compactum  $X$  such that every continuous map  $f : X \rightarrow Y$  has a dimensionally regular value.*

It still remains open whether every Boltyanskii compactum of dimension 5 admits a dimensionally exotic map. We note that the proof of Theorem 1.2 closely related to the construction by the first author of 4-dimensional ANR compacta  $X$  and  $Y$  with  $\dim(X \times Y) = 7$  [2],[6]. It is a long standing open problem whether such phenomenon can happen for 3-dimensional ANRs. Thus, our remaining problem in dimension 5 could be closely related to this one and quite difficult.

In the next section we briefly review basic facts of Cohomological Dimension and prove Theorem 1.3. Auxiliary constructions and propositions for proving Theorem 1.2 are presented in Section 3 and Theorem 1.2 is proved in the last section.

## 2 Cohomological Dimension

In this section we review basic facts of Cohomology Dimension and prove Theorem 1.3.

By cohomology we always mean the Čech cohomology. Let  $G$  be an abelian group. The **cohomological dimension**  $\dim_G X$  of a space  $X$  with respect to the coefficient group  $G$  does not exceed  $n$ ,  $\dim_G X \leq n$  if  $H^{n+1}(X, A; G) = 0$  for every closed  $A \subset X$ . We note that this condition implies that  $H^{n+k}(X, A; G) = 0$  for all  $k \geq 1$  [10],[6]. Thus,  $\dim_G X$  = the smallest integer  $n \geq 0$  satisfying  $\dim_G X \leq n$  and  $\dim_G X = \infty$  if such an integer does not exist. Clearly,  $\dim_G X \leq \dim_{\mathbb{Z}} X \leq \dim X$ .

**Theorem 2.1 (Alexandroff)**  $\dim X = \dim_{\mathbb{Z}} X$  if  $X$  is a finite dimensional space.

Let  $\mathcal{P}$  denote the set of all primes. The *Bockstein basis* is the collection of groups  $\sigma = \{\mathbb{Q}, \mathbb{Z}_p, \mathbb{Z}_{p^\infty}, \mathbb{Z}_{(p)} \mid p \in \mathcal{P}\}$  where  $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$  is the  $p$ -cyclic group,  $\mathbb{Z}_{p^\infty} = \varinjlim \mathbb{Z}_{p^k}$  is the  $p$ -adic circle, and  $\mathbb{Z}_{(p)} = \{m/n \mid n \text{ is not divisible by } p\} \subset \mathbb{Q}$  is the  $p$ -localization of integers.

The Bockstein basis of an abelian group  $G$  is the collection  $\sigma(G) \subset \sigma$  determined by the rule:

- $\mathbb{Z}_{(p)} \in \sigma(G)$  if  $G/\text{Tor}G$  is not divisible by  $p$ ;
- $\mathbb{Z}_p \in \sigma(G)$  if  $p\text{-Tor}G$  is not divisible by  $p$ ;
- $\mathbb{Z}_{p^\infty} \in \sigma(G)$  if  $p\text{-Tor}G \neq 0$  is divisible by  $p$ ;
- $\mathbb{Q} \in \sigma(G)$  if  $G/\text{Tor}G \neq 0$  is divisible by all  $p$ .

Thus  $\sigma(\mathbb{Z}) = \{\mathbb{Z}_{(p)} \mid p \in \mathcal{P}\}$ .

**Theorem 2.2 (Bockstein Theorem)** For a compactum  $X$ ,

$$\dim_G X = \sup\{\dim_H X : H \in \sigma(G)\}.$$

Suggested by the Bockstein inequalities we say that a function  $D : \sigma \rightarrow \mathbb{N} \cup \{0, \infty\}$  is a  $p$ -regular if  $D(\mathbb{Z}_{(p)}) = D(\mathbb{Z}_p) = D(\mathbb{Z}_{p^\infty}) = D(\mathbb{Q})$  and it is  $p$ -singular if  $D(\mathbb{Z}_{(p)}) = \max\{D(\mathbb{Q}), D(\mathbb{Z}_{p^\infty}) + 1\}$ . A  $p$ -singular function  $D$  is called  $p^+$ -singular if  $D(\mathbb{Z}_{p^\infty}) = D(\mathbb{Z}_p)$  and it is called  $p^-$ -singular if  $D(\mathbb{Z}_{p^\infty}) = D(\mathbb{Z}_p) - 1$ . A function  $D : \sigma \rightarrow \mathbb{N} \cup \{0, \infty\}$  is called a *dimension type* if for every prime  $p$  it is either  $p$ -regular or  $p^\pm$ -singular. Thus, the values of  $D(F)$  for the Bockstein fields  $F \in \{\mathbb{Z}_p, \mathbb{Q}\}$  together with  $p$ -singularity types of  $D$  determine the value  $D(G)$  for all groups in  $\sigma$ . For a dimension type  $D$  denote  $\dim D = \sup\{D(G) : G \in \sigma\}$ .

**Theorem 2.3 (Bockstein Inequalities [10], [6])** For every space  $X$  the function  $d_X : \sigma \rightarrow \mathbb{N} \cup \{0, \infty\}$  defined as  $d_X(G) = \dim_G X$  is a dimension type.

If  $X$  is compactum  $d_X$  is called the *dimension type* of  $X$ .

**Theorem 2.4 (Dranishnikov Realization Theorem [2],[3])** For every dimension type  $D$  there is a compactum  $X$  with  $d_X = D$  and  $\dim X = \dim D$ .

**Theorem 2.5 (Olszewski Completion Theorem [12])** For every space  $X$  there is a complete space  $X'$  such that  $X \subset X'$  and  $d_X = d_{X'}$ .

Let  $D$  be a dimension type. We will use the abbreviations  $D(0) = D(\mathbb{Q})$ ,  $D(p) = D(\mathbb{Z}_p)$ . Additionally, if  $D(p) = n \in \mathbb{N}$  we will write  $D(p) = n^+$  if  $D$  is  $p^+$ -regular and  $D(p) = n^-$  if it is  $p^-$ -regular. For a  $p$ -regular  $D$  we leave it without decoration:  $D(p) = n$ .

Thus, any sequence of decorated numbers  $D(p) \in \mathbb{N}$ , where  $p \in \mathcal{P} \cup \{0\}$  define a unique dimension type. There is a natural order on decorated numbers

$$\dots < n^- < n < n^+ < (n+1)^- < \dots$$

Note that the inequality of dimension types  $D \leq D'$  as functions on  $\sigma$  is equivalent to the family of inequalities  $D(p) \leq D'(p)$  for the above order for all  $p \in \mathcal{P} \cup \{0\}$ . The natural involution on decorated numbers that exchange the decorations '+' and '-' keeping the base fixed defines an involution  $*$  on the set of dimension types. Thus,  $*$  takes  $p^+$ -singular function  $D$  to  $p^-$ -singular  $D^*$  and vice versa.

Let  $D_1$  and  $D_2$  are dimension types. Suggested by Bockstein Product Theorem we define the dimension type  $D_1 \boxplus D_2$  as follows. If  $D_1(p) = n^{\epsilon_1}$  and  $D_2(p) = m^{\epsilon_2}$  where  $\epsilon_i$  is a decoration, i.e., '+' or '-' or empty, then

$$(D_1 \boxplus D_2)(p) = (n+m)^{\epsilon_1 \otimes \epsilon_2}$$

with the product of the signs  $\epsilon_1 \otimes \epsilon_2$  defined by

$$\epsilon \otimes \text{empty} = \epsilon, \quad \epsilon \otimes \epsilon = \epsilon, \quad \epsilon = \pm, \quad \text{and} \quad + \otimes - = -.$$

It turns out that  $D_1 \boxplus D_2$  is indeed a dimension type and its definition is justified by

**Theorem 2.6 (Bockstein Product Theorem [6],[13], [9])** *For any two compacta  $X$  and  $Y$*

$$d_{X \times Y} = d_X \boxplus d_Y.$$

By  $B_n$  we denote the dimension type such that  $B_n(p) = (n-1)^+$  for all  $p \in \mathcal{P}$  and  $B_n(\mathbb{Q}) = n-1$ . The product formula implies that an  $n$ -dimensional compactum  $X$  is a Boltyanskii compactum if and only if  $d_X \leq B_n$ .

For an integer  $n \geq 0$  we denote by  $n$  the dimension type which sends every  $G \in \sigma$  to  $n$ . For a dimension type  $D$  by  $D+n$  we mean the ordinary sum of  $D$  and  $n$  as functions. Note that  $D+n$  is also a dimension type.

For dimension types  $D_1$  and  $D_2$  we introduce another important operation

$$D_1 \oplus D_2 = (D_1^* \boxplus D_2^*)^*.$$

It can be shown that  $D_1 \oplus D_2$  is a dimension type as well. The operation  $\oplus$  is justified by the following properties.

**Theorem 2.7 (Dydak Union Theorem[8],[5])** *Let  $X$  be a compactum and  $D_1$  and  $D_2$  dimension types and let  $X = A \cup B$  be a decomposition with  $d_A \leq D_1$  and  $d_B \leq D_2$ . Then  $d_X \leq D_1 \oplus D_2 + 1$ .*

**Theorem 2.8 (Dranishnikov Decomposition Theorem [3], [5])** *Let  $X$  be a finite dimensional compactum and  $D_1$  and  $D_2$  dimension types such that  $d_X \leq D_1 \oplus D_2 + 1$ . Then there is a decomposition  $X = A \cup B$  such that  $d_A \leq D_1$  and  $d_B \leq D_2$ .*

**Theorem 2.9 (Levin-Lewis [11], [5])** *Let  $f : X \rightarrow Y$  be a map of compacta such that  $d_f \leq D_1$  and  $d_Y \leq D_2$  where  $d_f \leq D_1$  means that the dimension type of each fiber of  $f$  is less or equal  $D_1$ . Then  $d_X \leq D_1 \oplus D_2$ .*

Let  $D_1$  and  $D_2$  be dimension types. It can be shown that  $D_1 \boxplus D_2 \leq D_1 \oplus D_2$  and if  $D'_1$  and  $D'_2$  are dimension types such that  $D_1 \leq D'_1$  and  $D_2 \leq D'_2$  then  $D_1 \boxplus D_2 \leq D'_1 \boxplus D'_2$  and  $D_1 \oplus D_2 \leq D'_1 \oplus D'_2$ . Note that  $D \boxplus n = D \oplus n = D + n$  for any dimension type  $D$ .

An  $n$ -dimensional space  $X$  is said to be dimensionally full-valued if  $d_X = n$ . Note that every 0 or 1-dimensional compactum is dimensionally full-valued. P. S. Alexandroff proved the following dual statement:

**Theorem 2.10 ([10])** *Let  $X$  be an  $(n - 1)$ -dimensional compact subset of  $\mathbb{R}^n$ . Then  $X$  is dimensionally full-valued.*

Now we are ready to prove

**Theorem 2.11** *Let  $X$  be an  $n$ -dimensional Boltyanskii compactum,  $n \geq 4$ , containing an  $(n - 1)$ -dimensional cube. Then  $X$  does not admit a dimensionally exotic map  $f : X \rightarrow Y$  to a compactum  $Y$  with  $\dim Y \leq 2$ .*

**Proof.** Let  $B \subset X$  be an  $(n - 1)$ -dimensional cube. Aiming at a contradiction assume that there is a dimensionally exotic map  $f : X \rightarrow Y$  with  $\dim Y \leq 2$ . First notice that  $Y$  is not dimensionally full-valued since otherwise  $\dim(Y \times f^{-1}(y)) = \dim Y + \dim f^{-1}(y)$  and by Hurewicz's theorem  $f$  cannot be exotic. Thus we have  $\dim Y = 2$ . Denote  $k = \dim f|_B$ . Again by Hurewicz's theorem  $k \geq n - 3$ . If  $k = n - 3$  then by Theorem 2.9 we have  $n - 1 = d_B \leq d_Y \oplus (n - 3) = d_Y + (n - 3)$  and this contradicts to the fact that  $Y$  is 2-dimensional and not dimensionally full-valued. The case  $k = n - 1$  is also impossible because then there is a fiber  $F$  of  $f$  containing an  $(n - 1)$ -dimensional cube and hence  $\dim(Y \times F) = 2 + n - 1 = n + 1$  that contradicts the fact that  $f$  is exotic. So we are left with the only case  $k = n - 2$ . Then there is a fiber  $F$  of  $f$  with  $\dim F \cap B = n - 2$  and, by Proposition 2.10,  $d_F \geq n - 2$ . Thus  $d_{Y \times F} = d_Y \boxplus d_F \geq d_Y + (n - 2)$  and hence  $\dim(Y \times F) \geq \dim Y + n - 2 = n$  and this contradicts the assumption that  $f$  is dimensionally exotic. The theorem is proved. ■

**Proof of Theorem 1.3.** It follows immediately from Theorem 2.11. Indeed, take a 4-dimensional Boltyanskii compactum  $X$  containing a 3-dimensional cube and assume that  $f : X \rightarrow Y$  is a dimensionally exotic map to a compactum  $Y$ . Clearly  $\dim Y \leq 3$  and by Theorem 2.11 we have  $\dim Y = 3$ . Then by Hurewicz's theorem there is a fiber  $F$  of  $f$  with  $\dim F \geq 1$  and hence  $f$  cannot be exotic because  $\dim(Y \times F) \geq \dim Y + 1 = 4$ . ■

Note that Theorem 2.11 also implies that the dimension of  $Y$  in Theorem 1.2 cannot be reduced to 2. We don't know if it can be reduced to 3.

### 3 Auxiliary propositions and constructions

Let  $X \rightarrow Y$  be a map to a simplicial complex  $Y$  and  $A \subset X$ . We say that  $f$  is *dimensionally deficient* on  $A$  if  $\dim(f^{-1}(\Delta) \cap A) < \dim \Delta$  for every simplex  $\Delta$  in  $Y$ . The following two propositions are straightforward applications of the Baire category theorem and their proof is left to the reader.

**Proposition 3.1** *Let  $X$  be a compactum,  $M$  a triangulated  $n$ -dimensional manifold possibly with boundary,  $A$  a  $\sigma$ -compact subset of  $X$  such that  $\dim A \leq n - 1$ ,  $F$  a closed subset of  $X$ , and let  $f : X \rightarrow M$  be a map which is dimensionally deficient on  $A \cap F$ . Then the map  $f$  can be arbitrarily closely approximated by a map  $f' : X \rightarrow M$  such that  $f'$  is dimensionally deficient on  $A$  and  $f'$  coincides with  $f$  on  $F$ .*

**Proposition 3.2** *Let  $A \subset X$  be a  $\sigma$ -compact subset of a compactum  $X$ , and let  $f : X \rightarrow K$  be a map to a finite simplicial complex  $K$  such that  $f$  is dimensionally deficient on  $A$ . Then for every subdivision  $K'$  of  $K$  the map  $f$  can be arbitrarily approximated by a map  $f' : X \rightarrow K'$  such that  $f'$  is dimensionally deficient on  $A$  with respect to the triangulation of  $K'$  and for every simplex  $\Delta$  of  $K$  we have that  $f^{-1}(\Delta) = f'^{-1}(\Delta)$ .*

The following proposition besides the Baire category theorem uses the fact  $n + 1$   $(n - 1)$ -dimensional planes in  $\mathbb{R}^n$  in a general position do not intersect. This fact implies that a generic map  $f : Y \rightarrow \Delta^n$  of an  $(n - 1)$ -dimensional compactum to an  $n$ -dimensional simplex has  $|f^{-1}(x)| \leq n$  for all  $x \in \Delta^n$ .

**Proposition 3.3** *Let  $X$  be a compactum,  $A$  a  $\sigma$ -compact subset, and let  $f : X \rightarrow K$  be a map to a finite simplicial complex  $K$  such that  $f$  is dimensionally deficient on  $A$ . Then  $f$  can be arbitrarily closely approximated by a map  $f'$  such that for every simplex  $\Delta$  of  $K$  and for every  $y \in \Delta$  we have that  $f^{-1}(\Delta) = f'^{-1}(\Delta)$  and the number of points in  $f'^{-1}(y) \cap A$  does not exceed  $\dim \Delta$ . Clearly that  $f'$  is dimensionally deficient on  $A$  as well.*

We will also need

**Proposition 3.4** ([5]) *Let  $X$  be a finite dimensional compactum and  $n > 0$ . Then  $\dim_{\mathbb{Q}} X \leq n$  if and only if for every closed subset  $A$  of  $X$  and every map  $f : A \rightarrow S^n$  there is a map  $g : S^n \rightarrow S^n$  of non-zero degree such that  $g \circ f : X \rightarrow S^n$  continuously extends over  $X$ .*

Let  $g, h : X \rightarrow Y$  be maps. Consider the product  $X \times [0, 1]$  and consider  $g$  and  $h$  as the maps from  $X \times \{0\}$  and  $X \times \{1\}$  respectively. We recall that the *double mapping cylinder*  $M(g, h)$  of  $g$  and  $h$  is the quotient space of  $X \times [0, 1]$  in which  $X \times \{0\}$  is identified with  $Y$  according to the map  $g$  and  $X \times \{1\}$  is identified with another copy of  $Y$  according to the map  $h$ . The quotient map  $P : X \times [0, 1] \rightarrow M(g, h)$  will be called the *cylinder projection* and the map  $p : M(g, h) \rightarrow [0, 1]$  induced by the map  $(x, t) \rightarrow t$

will be called the *interval projection*. We call  $p^{-1}(0)$  and  $p^{-1}(1)$  the  $g$ -part and the  $h$ -part of  $M(g, h)$  respectively. For pointed maps  $g, h : (X, x_0) \rightarrow (Y, y_0)$  we identify the interval  $[0, 1]$  with the interval  $P(\{x_0\} \times [0, 1])$  and call it the *axis* of  $M(g, h)$ . We can rescale or shift the interval  $[0, 1]$  to an interval  $[a, b]$  and in that case we say that  $M(g, h)$  is the cylinder over  $[a, b]$ . Let  $g_i, h_i : X \rightarrow Y, 1 \leq i \leq k$  be maps. By the *telescope*  $M((g_1, h_1), \dots, (g_k, h_k))$  we mean the union of the cylinders  $M(g_i, h_i)$  where the  $h_i$ -part of  $M(g_i, h_i)$  is identified with the  $g_{i+1}$ -part of  $M(g_{i+1}, h_{i+1})$  for every  $i$ . Partition  $[0, 1]$  into  $0 = t_0 < t_1 < \dots < t_k = 1$  and consider  $M(g_i, h_i)$  as the cylinder over  $[t_{i-1}, t_i]$ . Then the interval projections of the cylinders define the interval projection  $p$  from the telescope  $M((g_1, h_1), \dots, (g_k, h_k))$  to the interval  $[0, 1]$  such that  $M(g_i, h_i) = p^{-1}([t_{i-1}, t_i])$ . For pointed maps the interval  $[0, 1]$  is identified with the interval in the telescope so that  $[t_{i-1}, t_i]$  is the axis of the cylinder  $M(g_i, h_i)$ . After such an identification we refer to  $[0, 1]$  as the telescope axis. To shorten the notation for  $M((g_1, h_1), \dots, (g_k, h_k))$  we denote  $\Pi = \{(g_1, h_1), \dots, (g_k, h_k)\}$  and then denote the telescope by  $M(\Pi)$ . A space is said to be a *PL-complex* if it admits a triangulation which determines its PL-structure. If  $X$  and  $Y$  are PL-complexes and the maps in  $\Pi$  are PL-maps then we always consider  $M(\Pi)$  with an induced PL-structure for which each cylinder projection  $P_i : X \times [t_{i-1}, t_i] \rightarrow M(g_i, h_i)$  and the interval projection  $p : M(\Pi) \rightarrow [0, 1]$  are PL-maps.

Let  $B$  be an  $l$ -dimensional ball with center  $O$ . We consider  $B$  as a pointed space with  $O$  as the base point. By a *simple map*  $f : B \rightarrow B$  we mean a map which radially extends a finite-to-one PL-map from  $\partial B$  to  $\partial B$ . More precisely, we consider  $B$  as the cone over  $\partial B$  with the vertex at the center  $O$ . Then a simple map is the cone of a map from  $\partial B$  to  $\partial B$ . For a simple map  $f : B \rightarrow B$  we denote by  $\partial f$  the map  $\partial f : \partial B \rightarrow \partial B$  which is the restriction of  $f$  to  $\partial B$  and by the degree of a simple map  $f : B \rightarrow B$  we mean the degree of  $\partial f$ . We say that a simple map is *non-degenerate* if it has a non-zero degree. Clearly simple maps preserve the base point of  $B$ . Let  $g, h : B \rightarrow B$  be simple maps. We consider the cylinder  $M(\partial g, \partial h)$  as embedded in  $M(g, h)$  and we will call  $M(\partial g, \partial h)$  the boundary cylinder of  $M(g, h)$ . For a collection  $\Pi = \{(g_1, h_1), \dots, (g_k, h_k)\}$  of pairs of simple maps from  $B$  to  $B$  we denote  $\partial \Pi = \{(\partial g_1, \partial h_1), \dots, (\partial g_k, \partial h_k)\}$  and we call  $M(\partial \Pi) \subset M(\Pi)$  the *boundary telescope* of  $M(\Pi)$ .

Let  $f : X \rightarrow Y$  be a PL-map of PL-complexes. A point  $x \in X$  is said to be a *simple point of degree  $d$*  of the map  $f$  if there are closed PL-neighborhoods  $X'$  and  $Y'$  of  $x$  and  $y = f(x)$  respectively such that  $f(X') = Y', X' = f^{-1}(Y')$  and there are PL-homeomorphisms from  $X'$  and  $Y'$  to a ball  $B$  sending  $x$  and  $y$  to the center  $O$  of  $B$  such that  $f$  restricted to  $X'$  and  $Y'$  acts as a simple map of  $B$  of degree  $d$ . We say that a simple point is *non-degenerate* if it has non-zero degree. Note that if  $f : S^n \rightarrow S^n$  has a simple point of degree  $d$  then  $\deg f = d$ . Also note that if we have a collection of disjoint  $n$ -balls  $B_1, \dots, B_m \subset S^n$  which are PL-embedded in  $S^n, n \geq 3$ , and a collection of simple maps  $f_i : B_i \rightarrow B_i$  such that  $\deg f_i = d$  for every  $i$  then there is a finite-to-one PL-map  $f : S^n \rightarrow S^n$  so that  $f$  extends  $f_i$  and  $f^{-1}(B_i) = B_i$  for every  $i$  and, as noted before,

$\deg f = d$ .

Consider a telescope  $M(\Pi)$  of a collection  $\Pi = \{(g_1, h_1), \dots, (g_k, h_k)\}$  and of non-degenerate simple maps of an  $l$ -ball  $B$ ,  $l \geq 3$  and suppose that  $[0, 1]$  is partitioned into  $0 = t_0 < t_1 < \dots < t_k = 1$  such that  $M(g_i, h_i) = p^{-1}([t_{i-1}, t_i])$  where  $p : M(\Pi) \rightarrow [0, 1]$  is the interval projection.

**Proposition 3.5** *For every map  $\phi : X \rightarrow M(\Pi)$  from a space  $X$  with  $\dim_{\mathbb{Q}} X \leq l$  there is a finite-to-one PL-map  $\psi : M(\partial\Pi) \rightarrow \partial B \times [0, 1]$  such that*

- $\psi^{-1}(\partial B \times [t_{i-1}, t_i]) = M(\partial g_i, \partial h_i)$ ,  $1 \leq i \leq k$ ;
- for every  $1 \leq i \leq k$  the map  $\psi$  has non-degenerate simple points  $z'_i, z''_i \in M(\partial g_i, \partial h_i)$  such that  $z'_i \neq z''_i$ ,  $p(z'), p(z'') \in (t_{i-1}, t_i)$  and  $\psi(z'_i), \psi(z''_i) \in \partial B \times (t_{i-1}, t_i)$ ;
- $\phi$  restricted to  $\phi^{-1}(M(\partial\Pi))$  and followed by  $\psi$  extends over  $X$  to a map  $\phi' : X \rightarrow \partial B \times [0, 1]$  such that  $\phi'^{-1}(\partial B \times ([t_{i-1}, t_i])) = \phi^{-1}(M(g_i, h_i))$ ,  $1 \leq i \leq k$ .

**Proof.** Take simple PL-maps  $f_i : B \rightarrow B$ ,  $1 \leq i \leq k+1$  such that  $\deg f_i + \deg g_i = \deg f_{i+1} + \deg h_i$  and assuming that  $\deg f_1$  is sufficiently large we may also assume that  $\deg f_i + \deg g_i = \deg f_{i+1} + \deg h_i \neq 0$ . Denote  $g'_i = f_i \circ g_i$ ,  $h'_i = f_{i+1} \circ h_i$ ,  $1 \leq i \leq k$ , and  $\Pi' = \{(g'_1, h'_1), \dots, (g'_k, h'_k)\}$ . Then there is a natural projection  $\pi : M(\Pi) \rightarrow M(\Pi')$  and it is easy to see that we can replace  $\Pi$  by  $\Pi'$  and  $\phi$  by  $\pi \circ \phi$  and assume that  $\deg g_i = \deg h_i$  for every  $1 \leq i \leq k$ .

Fix points  $z'_i \neq z''_i, z'_i, z''_i \in \partial B \times (t_{i-1}, t_i)$ . Consider a finite-to-one PL-map  $\alpha_i : \partial(B \times [t_{i-1}, t_i]) \rightarrow \partial(B \times [t_{i-1}, t_i])$  so that  $\alpha_i$  coincides with  $g_i$  on  $B \times \{t_{i-1}\}$ , with  $h_i$  on  $B \times \{t_i\}$ ,  $\alpha_i^{-1}(B \times \{t_{i-1}\}) = B \times \{t_{i-1}\}$ ,  $\alpha_i^{-1}(B \times \{t_i\}) = B \times \{t_i\}$ ,  $z'_i$  and  $z''_i$  are simple points of  $\alpha_i$  of  $\deg = \deg g_i = \deg h_i$  and  $\alpha_i(z'_i) = z'_i$  and  $\alpha_i(z''_i) = z''_i$ . Such a map  $\alpha_i$  exists because  $\partial(B \times [t_{i-1}, t_i])$  is a sphere of  $\dim \geq 3$ . Extend  $\alpha_i$  to a finite-to-one PL-map  $\alpha'_i : B \times [t_{i-1}, t_i] \rightarrow B \times [t_{i-1}, t_i]$  so that  $\alpha'^{-1}_i(\partial(B \times [t_{i-1}, t_i])) \subset \partial(B \times [t_{i-1}, t_i])$ . Then the cylinder projection  $P_i : B \times [t_{i-1}, t_i] \rightarrow M(g_i, h_i)$  factors through  $\alpha'_i$  and a finite-to-one PL-map  $\alpha''_i : M(g_i, h_i) \rightarrow B \times [t_{i-1}, t_i]$  so that  $\alpha''_i$  sends the  $g_i$ -part and the  $h_i$ -part of  $M(g_i, h_i)$  by the identity maps to  $B \times \{t_{i-1}\}$  and  $B \times \{t_i\}$  respectively. The maps  $\alpha''_i$  define a map  $\beta : M(\Pi) \rightarrow B \times [0, 1]$ . We will also denote by  $z'_i$  and  $z''_i$  the images  $P_i(z'_i)$  and  $P_i(z''_i)$  of  $z'_i$  and  $z''_i$  respectively in  $M(g_i, h_i)$  under the cylinder projection. Thus we have that the points  $z'_i$  and  $z''_i$  are non-degenerate simple points of  $\beta$  restricted to  $M(\partial\Pi)$  and  $\partial B \times [0, 1]$ . Denote  $\phi_0 = \beta \circ \phi : X \rightarrow B \times [0, 1]$ .

We will construct for every  $0 \leq j \leq k$  a map  $\phi_j : X \rightarrow B \times [0, 1]$  and a finite-to-one PL-map  $\beta_j : B \times [0, 1] \rightarrow B \times [0, 1]$  such that for  $1 \leq i \leq k$  we have that  $\phi_{j+1}$  and  $\beta_{j+1} \circ \phi_j$  coincide on  $\phi_j^{-1}(\partial B \times [0, 1])$ ,  $\phi_{j+1}^{-1}(B \times [t_{i-1}, t_i]) = \phi_j^{-1}(B \times [t_{i-1}, t_i])$ ,  $0 \leq j \leq k-1$ ,  $\phi_j(X) \subset \partial B \times [t_0, t_j] \cup B \times [t_j, t_k]$ ,  $\beta_j^{-1}(B \times [t_{i-1}, t_i]) \subset B \times [t_{i-1}, t_i]$ ,  $\beta_j^{-1}(\partial B \times [t_{i-1}, t_i]) \subset \partial B \times [t_{i-1}, t_i]$ ,  $\beta_j^{-1}(B \times \{0\}) \subset B \times \{0\}$ ,  $\beta_j^{-1}(B \times \{1\}) \subset B \times \{1\}$ ,  $z'_i$  and  $z''_i$  are non-degenerate simple points of  $\beta_j$  restricted to  $\partial B \times [0, 1]$ , and  $z'_i$  and  $z''_i$



are also fixed points of  $\beta_j$ . Set  $\beta_0 = id$ , assume that the construction is completed for  $j$  and proceed to  $j + 1$  as follows.

Denote  $X' = \phi_j^{-1}(B \times \{t_j\})$  and  $X'' = \phi_j^{-1}(\partial B \times \{t_j\})$ . By Proposition 3.4 there is a map  $g : B \rightarrow B$  of non-zero-degree such that if we consider  $g$  as a map of  $B \times \{t_j\}$  then the map  $\phi_j$  restricted to  $X''$  and followed by  $g$  continuously extends to  $\Phi' : X' \rightarrow \partial B \times \{t_j\}$ . Once again for every  $1 \leq i \leq k$  take a finite-to-one PL-map  $\alpha_i : \partial(B \times [t_{i-1}, t_i]) \rightarrow \partial(B \times [t_{i-1}, t_i])$  so that  $\alpha_i$  coincides with  $g$  on  $B \times \{t_{i-1}\}$  and on  $B \times \{t_i\}$ ,  $\alpha_i^{-1}(B \times \{t_{i-1}\}) = B \times \{t_{i-1}\}$ ,  $\alpha_i^{-1}(B \times \{t_i\}) = B \times \{t_i\}$  and  $\alpha_i(z'_i) = z'_i$ ,  $\alpha_i(z''_i) = z''_i$  and  $z'_i$  and  $z''_i$  are simple points of  $\alpha_i$  of  $\deg g$ . Extend  $\alpha_i$  to a finite-to-one PL-map  $\alpha'_i : B \times [t_{i-1}, t_i] \rightarrow B \times [t_{i-1}, t_i]$  so that  $\alpha'^{-1}_i(\partial(B \times [t_{i-1}, t_i])) \subset \partial(B \times [t_{i-1}, t_i])$ . and denote by  $\beta_{j+1} : B \times [0, 1] \rightarrow B \times [0, 1]$  the map defined by  $\alpha'_i$ ,  $1 \leq i \leq k$ . Let  $\Phi'' = \beta_{j+1} \circ \phi_j : X \rightarrow B \times [0, 1]$ . Recall that  $\Phi''$  restricted to  $X''$  extends to a map  $\Phi' : X' \rightarrow \partial B \times \{t_j\}$ . Since  $Y = \partial B \times [t_j, t_{j+1}] \cup B \times \{t_{j+1}\}$  is contractible we can extend  $\Phi''$  restricted to  $\Phi''^{-1}(Y)$  to a map  $\Phi''' : \Phi''^{-1}(B \times [t_j, t_{j+1}]) \rightarrow Y$  so that  $\Phi'''$  coincides with  $\Phi'$  on  $X'$  and with  $\Phi''$  on  $\Phi''^{-1}(Y)$ . Thus we define a map  $\phi_{j+1} : X \rightarrow B \times [0, 1]$  by changing  $\Phi''$  on  $\Phi''^{-1}(B \times [t_j, t_{j+1}])$  according to  $\Phi'''$  and get that  $\phi_{j+1}(X) \subset \partial B \times [t_0, t_{j+1}] \cup B \times [t_{j+1}, t_k]$ . It is easy to see that all the properties of  $\phi_{j+1}$  and  $\beta_{j+1}$  that we required are satisfied.

Thus we finally construct a map  $\phi_k$  with  $\phi_k(X) \subset \partial B \times [0, 1] \cup B \times \{t_k\}$ . It is obvious that the construction of  $\phi_{j+1}$  and  $\beta_{j+1}$  also applies to construct from  $\phi_k$  a map  $\phi_{k+1} : X \rightarrow B \times [0, 1]$  and a map  $\beta_{k+1}$  so that all the properties of  $\phi_j$  and  $\beta_j$  will be satisfied except the one regarding  $\phi_j(X)$  which we change to  $\phi_{k+1}(X) \subset \partial B \times [0, 1]$ . Set  $\phi' = \phi_{k+1}$ ,  $\psi = \beta_{k+1} \circ \dots \circ \beta_0 \circ \beta|_{M(\partial\Pi)}$ . Clearly the construction can be carried out so that  $z'_i$  and  $z''_i$  will also be simple points of  $\psi$  and the proof is completed. ■

Let  $Z$  be a CW-complex with  $\dim Z = l + 1$ . Assume that  $Z$  also has a PL-structure which agrees with the CW-structure. By this we mean that for every closed cell  $C$  of  $Z$  both  $C$  and  $\partial C$  are PL-subcomplexes of  $Z$ . A cylinder  $M(g, h)$  of non-degenerate simple maps of an  $l$ -ball  $B$  is said to be *properly embedded* in a closed  $(l + 1)$ -cell  $C$  of  $Z$  if  $M(g, h) \subset C$  is a PL-embedding into  $C$ ,  $\partial C \cap M(g, h) =$  the union of the  $g$ -part and the  $h$ -part of  $C$  and  $M(g, h) \setminus M(\partial g, \partial h)$  is open in  $C$ . A telescope  $M(\Pi)$  of a collection  $\Pi = \{(g_1, h_1), \dots, (g_k, h_k)\}$  of non-degenerate simple maps of an  $l$ -ball  $B$  is said to be *properly embedded* into  $Z$  if there are closed  $(l + 1)$ -cells  $C_1, \dots, C_k$  of  $Z$  such that each cylinder  $M(g_i, h_i)$  is properly embedded into  $C_i$ ,  $1 \leq i \leq k$ , and  $M(\Pi) \setminus M(\partial\Pi)$  is open in  $Z$ . An embedding  $\gamma : [0, 1] \rightarrow Z$  is said to be a *proper path* from  $z_0 = \gamma(0)$  to  $z_1 = \gamma(1)$  if the path  $\gamma$  identified with  $[0, 1]$  is the axis of a telescope properly embedded in  $Z$ . We always assume that a proper path  $\gamma$  is parametrized by the interval projection  $p$  of a telescope witnessing that  $\gamma$  is proper, that is  $p(\gamma(t)) = t$  for every  $t \in [0, 1]$ . It is easy to see that for a proper path  $\gamma$  we can choose a telescope  $M(\Pi) \subset Z$  witnessing that  $\gamma$  is a proper path to be so close to the path  $\gamma$  that the interval projection  $p : M(\Pi) \rightarrow [0, 1]$

to the axis will have the diameter of the fibers as small as we wish.

## 4 Proof of Theorem 1.2

Let  $X$  be an  $n$ -dimensional Boltyanskii compactum,  $n \geq 6$ . Recall that  $d_X \leq B_n$ . Consider the dimension types  $D_1$  and  $D_2$  defined for every prime  $p$  by:

$$D_1(p) = 3^-, \quad D_1(\mathbb{Q}) = 2, \quad \text{and} \quad D_2(p) = (n-5)^+, \quad D_2(\mathbb{Q}) = n-4.$$

Note that  $B_n = D_1 \oplus D_2 + 1$  and  $\dim(D_1 + 1) \boxplus D_2 = n - 1$ . By Theorem 2.8 there is a decomposition  $X = A \cup B$  of  $X$  such that  $d_A \leq D_1$  and  $d_B \leq D_2$ . We may assume that  $A = X \setminus B$  and, by Theorem 2.5, we may also assume that  $B$  is  $G_\delta$  and  $A$  is  $\sigma$ -compact. Represent  $A = \cup A_i$  as a countable union of compact sets  $A_i$  such that  $A_i \subset A_{i+1}$ .

We will construct for each  $i$  a 4-dimensional compact PL-complex  $Y_i$ , a bonding map  $\omega_i^{i+1} : Y_{i+1} \longrightarrow Y_i$  and a map  $\phi_i : X \longrightarrow Y_i$ . We fix metrics in  $X$  and in each  $Y_i$  and with respect to these metrics we determine  $0 < \epsilon_i < 1/2^i$  such that the following properties will be satisfied:

(i) for every open set  $U \subset Y_i$  with  $\text{diam} U < 2\epsilon_i$  the set  $\phi_i^{-1}(U) \cap A_i$  splits into at most four disjoint sets open in  $A_i$  and of  $\text{diam} \leq 1/i$ ;

(ii)  $\text{dist}(\omega_j^{i+1} \circ \phi_{i+1}, \omega_j^i \circ \phi_i) < \epsilon_j/2^i$  for  $i > j$  and  $\text{dist}(\omega_i^{i+1} \circ \phi_{i+1}|_{A_i}, \omega_i^i \circ \phi_i|_{A_i}) < \epsilon_i/2^i$  where  $\omega_i^j = \omega_{j-1}^j \circ \dots \circ \omega_i^{i+1} : Y_j \longrightarrow Y_i$  for  $j > i$  and  $\omega_i^i = \text{id} : Y_i \longrightarrow Y_i$ .

The construction will be carried out so that for  $Y = \text{invlim}(Y_i, \omega_i^{i+1})$  we have  $\dim_{\mathbb{Q}} Y \leq 3$ . Let us first show that the theorem follows from this construction. Denote  $f_i = \lim_{j \rightarrow \infty} \omega_i^j \circ \phi_j : X \longrightarrow Y_i$ . From (ii) it follows that  $f_i$  is well-defined, continuous and  $\text{dist}(f_i, \phi_i) \leq \epsilon_i$  on  $A_i$ . From the definition of  $f_i$  it follows that  $f_j \circ f_i^j = f_i$ . Hence the maps  $f_i$  define the corresponding map  $f : X \longrightarrow Y$  such that  $\omega_i \circ f = f_i$  where  $\omega_i : Y \longrightarrow Y_i$  is the projection. Then it follows from (i) that for every  $y \in Y_i$  the set  $f_i^{-1}(y) \cap A_i$  splits into at most four disjoint sets closed in  $A_i$  and of  $\text{diam} \leq 1/i$ . This implies that for every  $y \in Y$  we have that  $f^{-1}(y) \cap A_i$  contains at most 4 points and hence  $f^{-1}(y) \cap A$  contains at most 4-points and therefore  $d_f \leq D_2$ . Since  $\dim Y_i \leq 4$  we have  $\dim Y \leq 4$  and since  $\dim D_2 = n - 4$ , Hurewicz Theorem implies that  $\dim Y = 4$ . The condition  $\dim_{\mathbb{Q}} Y \leq 3$  implies that  $d_Y(p) \leq 4^-$  for all  $p$ . Therefore,  $d_Y \leq D_1 + 1$  and the theorem follows.

Let us begin the construction of  $Y_i$ ,  $\omega_i^{i+1}$  and  $\phi_i$ . In addition to (i) and (ii) we need a few more conditions to be satisfied. First we require that

(iii)  $\phi_i$  is dimensionally deficient on  $A$  for every  $i$ . By this we mean that there is triangulation of  $Y_i$  which agrees with the PL-structure of  $Y_i$  for which  $\phi_i$  is dimensionally deficient on  $A$ .

We will endow each  $Y_i$  with a structure of a CW-complex which agrees with the PL-structure of  $Y_i$ . For every  $i$  we will also construct a finite collection  $\Gamma_i$  of disjoint proper paths in  $Y_i$  and a finite closed cover  $\mathcal{F}_i$  of  $Y_i$  having the following properties:

(iv) for each  $i$  and each closed 4-cell  $C$  in  $Y_i$  there is a path  $\gamma \in \Gamma_i$  such that  $\gamma$  crosses  $C$  (we say that a path crosses a closed cell  $C$  if it passes through  $C \setminus \partial C$ );

(v) for each path  $\gamma \in \Gamma_i$  we have that the union of the closed 4-cells of  $Y_i$  that  $\gamma$  crosses is contained in a set of  $\mathcal{F}_i$  and the diameters of the sets in  $\omega_j^i(\mathcal{F}_i)$  is less than  $\epsilon_j/2^i$  for  $i > j$ .

Let  $Y_1$  be a 4-dimensional simplex. By Proposition 3.3, take a map  $\phi_1 : X \rightarrow Y_1$  such that  $\phi_1$  is 4-to-1 on  $A_1$ . Set  $\epsilon_1$  to satisfy (i). Take a triangulation of  $Y_1$  into small simplexes, set the CW-structure of  $Y_1$  to coincide with this triangulation and let  $\Gamma_1$  contain only one proper path  $\gamma$  to a boundary point of  $Y_1$  crossing each simplex of  $Y_1$ . A telescope witnessing that  $\gamma$  is an proper path is very simple, it is a telescope of the identity maps of  $B^3$ . We assume that the simplexes of  $Y_1$  are so small that (iv) is satisfied. By Propositions 3.1 and 3.3, replace  $\phi_1 : X \rightarrow Y_1$  by a map dimensionally deficient on  $A$  and so close to  $\phi_1$  that (i) remains true. Set  $\mathcal{F}_1 = \{Y_1\}$ . Assume that the construction is completed for  $i$  and proceed to  $i + 1$  as follows.

Fix a number  $\epsilon > 0$  which will be determined later. We are going to refine the CW-structure of  $Y_i$  and modify the paths in  $\Gamma_i$ . Take a closed 4-cell  $C$  of  $Y_i$  and a path  $\gamma \in \Gamma_i$  that  $\gamma$  crosses  $C$ . Let  $M(\Pi)$  be a telescope witnessing that  $\gamma$  is a proper path and let  $\gamma$  cross  $\partial C$  at the points  $y'_\gamma$  and  $y''_\gamma$  with  $t' = p(y'_\gamma)$ ,  $t'' = p(y''_\gamma)$  and  $t' < t''$ . Consider disjoint small closed neighborhoods of  $y'_\gamma$  and  $y''_\gamma$  of the form  $C'_\gamma = p^{-1}([t', t' + \delta])$  and  $C''_\gamma = p^{-1}([t'' - \delta, t''])$  respectively. Then the neighborhoods  $C'_\gamma$  and  $C''_\gamma$  are PL-subcomplexes of  $Y_i$  and taking  $M(\Pi)$  sufficiently close to  $\gamma$  we may assume that  $C'_\gamma$  and  $C''_\gamma$  are disjoint for distinct paths in  $\Gamma_i$ . Take a triangulation of  $Y_i$  underlying  $C'_\gamma$  and  $C''_\gamma$  for all paths  $\gamma \in \Gamma_i$  crossing  $C$ . Define the new closed 4-cells covering  $C$  to be  $C'_\gamma$  and  $C''_\gamma$  for all paths  $\gamma \in \Gamma_i$  crossing  $C$  and 4-simplexes not lying in the cells  $C'_\gamma$  and  $C''_\gamma$  we already defined. The cells of  $\dim \leq 3$  will be the simplexes of  $Y_i$  of  $\dim \leq 3$ . It is easy to see that each path  $\gamma$  in  $\Gamma_i$  crossing  $C$  can be modified on  $C$  so that it will cross every new closed 4-cell lying in  $C$ , will be proper with respect to the new cells and the modified paths of  $\Gamma_i$  will be disjoint. This can be done independently for every old closed 4-cell  $C$  of  $Y_i$ . Thus replacing the old CW-structure of  $Y_i$  and the old collection  $\Gamma_i$  by the new ones we may assume that every cell of  $Y_i$  is of  $\text{diam} < \epsilon$  and (iv) and (v) remain true with the cover  $\mathcal{F}_i$  left unchanged.

Take a path  $\gamma \in \Gamma_i$  and let  $M(\Pi)$  be a telescope of non-degenerate simple maps of a 3-ball  $B$  witnessing that  $\gamma$  is a proper path. Then the interval projection to the axis of  $M(\Pi)$  has fibers of  $\text{diam} < \epsilon$ . Denote  $X' = A_i \cap \phi_i^{-1}(M(\Pi))$  and  $\phi = \phi_i|_{X'}$ . By Proposition 3.5 there

are a finite-to-one PL-map  $\psi : M(\partial\Pi) \longrightarrow \partial B \times [0, 1]$  and a map  $\phi' : X' \longrightarrow \partial B \times [0, 1]$  satisfying the conclusions of 3.5 for  $X$  being replaced by  $X'$ . Define  $Y_{i+1}$  to be the quotient space of the disjoint union of  $Y'_i = (Y_i \setminus M(\Pi)) \cup M(\partial\Pi)$  and  $B \times [0, 1]$  obtained by identifying the points of  $M(\partial\Pi)$  with  $\partial B \times [0, 1]$  by the map  $\psi$  and let  $q_i : Y'_i \cup B \times [0, 1] \longrightarrow Y_{i+1}$  be the quotient map. Note that  $B \times [0, 1]$  is PL-homeomorphic to  $(\partial B \times [0, 1] \cup B \times \{0\}) \times [0, 1]$  and consider the retraction  $B \times [0, 1] \longrightarrow \partial B \times [0, 1] \cup B \times \{0\}$  coming from the projection to the first factor in the second representation of  $B \times [0, 1]$ . This retraction defines the corresponding retraction  $r_{i+1} : Y_{i+1} \longrightarrow Y_{i+1}^r = q_i(Y'_i) \cup B \times \{0\} \subset Y_{i+1}$ .

Define a map  $\phi_{i+1} : X \longrightarrow Y_{i+1}^r$  on  $X \setminus \phi_i^{-1}(M(\Pi) \setminus M(\partial\Pi))$  as the map  $\phi_i$  followed by  $q_i$ , on  $X'$  by  $\psi$  and on  $\phi_i^{-1}(M(\Pi) \setminus M(\partial\Pi))$  as any map to  $\partial B \times [0, 1] \cup B \times \{0\}$  extending what we already defined. Since  $\partial B \times [0, 1] \cup B \times \{0\}$  is homeomorphic to  $B$  such an extension exists.

Note that by Proposition 3.5 we have that the sets  $\partial B \times \{t\}, 0 \leq t \leq 1$ , each fiber of  $q_i$  and the sets  $\phi_{i+1}(\phi_i^{-1}(y)), y \in Y_i'' = A_i \cup (Y_i \setminus (M(\Pi) \setminus M(\partial\Pi)))$  are contained in closed 4-cells of  $Y_i$ . Hence taking  $\epsilon$  sufficiently small we can define a map  $\omega_{i+1}^r : Y_{i+1}^r \longrightarrow Y_i$  such that  $\omega_{i+1}^r \circ q_i|_{Y'_i}$  will be as close to the identity embedding of  $Y'_i$  into  $Y_i$  as we wish,  $\omega_{i+1}^r(\partial B \times [0, 1])$  will be as close to the path  $\gamma$  as we wish and the variation of  $\omega_{i+1}^r$  on the sets  $B \times \{0\}, \partial B \times \{t\}, 0 \leq t \leq 1$  and  $\phi_i(\phi_{i+1}^{-1}(y)), y \in Y_i''$  will be as small as we wish where the variation means the supremum of the diameters of the images of the sets under  $\omega_{i+1}^r$ . Set  $\omega_i^{i+1} = \omega_{i+1}^r \circ r_{i+1} : Y_{i+1} \longrightarrow Y_i$ . Then we may assume that the variation of  $\phi_i$  on the fibers of  $\omega_i^{i+1} \circ \phi_{i+1}$  over  $Y_i''$  is as small as we wish and  $\phi_i(\phi_i^{-1}(M(\Pi)))$  is contained in any given neighborhood of a set of  $\mathcal{F}_i$  containing all the cells of  $Y_i$  that  $\gamma$  passes through. Thus we may assume that (ii) holds.

Recall that, by Proposition 3.5,  $\psi$  is a finite-to-one PL-map. Then we define a PL-structure on  $Y_{i+1}$  for which  $q_i$  is a PL-map. Clearly that  $q_i$  is finite-to-one and, since  $\phi_i$  is dimensionally deficient on  $A$ , we get, by Proposition 3.2, that  $\phi_{i+1}$  can be replaced by an arbitrarily closed map which is dimensionally deficient on  $A \cap (Y_i \setminus (\phi_i^{-1}(M(\Pi) \setminus M(\partial\Pi))))$ . Then, by Proposition 3.1,  $\phi_{i+1}$  can be again arbitrarily closely approximated by a map which coincides with  $\phi_{i+1}$  on  $Y_i \setminus (\phi_i^{-1}(M(\Pi) \setminus M(\partial\Pi)))$  and dimensionally deficient on  $A \cap \phi_i^{-1}(M(\Pi))$ . Thus replacing  $\phi_{i+1}$  by such a map we may assume that (iii) holds. By Proposition 3.3 we may assume in addition that  $\phi_{i+1}$  is 4-to-1 on  $A_{i+1}$  and determine  $\epsilon_{i+1} > 0$  for which (i) holds.

Since the paths of  $\Gamma_i$  are disjoint we can take telescopes witnessing that the paths in  $\Gamma_i$  are proper to be disjoint and very close to the paths of  $\Gamma_i$ . Then the above construction can be carried out simultaneously for all the paths  $\gamma \in \Gamma_i$  and therefore we may assume that constructing  $Y_{i+1}$ ,  $\phi_{i+1}$  and  $\omega_i^{i+1}$  we took care of all the paths of  $\Gamma_i$ .

Now we will define a CW-structure of  $Y_{i+1}^r$ , a finite collection  $\Gamma_{i+1}^r$  of disjoint proper paths

in  $Y_{i+1}^r$  and establish a property that will be used in proving that  $\dim_{\mathbb{Q}} Y \leq 3$ . Let  $C$  be a closed 4-cell of  $Y_i$ .  $\gamma \in \Gamma_i$  a path that crosses  $C$  and  $M(\Pi)$  be the telescope witnessing that  $\gamma$  is a proper path that we used in the construction. Consider the cylinder  $M(g, h)$  of the telescope  $M(\Pi)$  such that  $M(g, h)$  lies in  $C$ . Assume that  $M(g, h)$  is a cylinder over the interval  $[t', t'']$  and recall that Proposition 3.5 applied  $M(\Pi)$  produced a map  $\psi$  with distinct non-degenerate simple points  $z'$  and  $z''$  in  $M(\partial g, \partial h)$  such that  $p(z'), p(z'') \in (t', t'')$ . Note that  $\psi$  induces the map from the sphere  $M(\partial g, \partial h)/(M(\partial g, \partial h) \cap \partial C)$  to the sphere  $\partial B \times [t', t'']/(\partial B \times \{t', t''\})$  and, since  $z'$  and  $z''$  are non-degenerate simple points of this induced map, its degree is equal to the degree of these points and hence different from 0. This implies that

$$(*) \ H^4(q_i(C \cap Y_i'), q_i(\partial C \cap Y_i'); \mathbb{Q}) = 0.$$

Denote  $C_r = q_i(C \cap Y_i')$ ,  $z'_r = \psi(z')$  and  $z''_r = \psi(z'')$  and consider  $z'_r$  and  $z''_r$  as points of  $Y_i^r$ . Then  $z'_r, z''_r$  have closed neighborhoods  $C'_r$  and  $C''_r$  in  $C_r$  of the form  $C'_r = M(g', id)$  and  $C''_r = M(g'', id)$  such that  $C'_r \cap C''_r = \emptyset$ ,  $C'_r$  and  $C''_r$  are disjoint for distinct paths in  $\Gamma_i$  that cross  $C$ ,  $g'$  and  $g''$  are simple non-degenerate maps of a 3-ball  $B$ ,  $M(g', id)$  and  $M(g'', id)$  are PL-embedded in  $C \setminus (M(\Pi) \setminus M(\partial(\Pi)))$ ,  $z'_r$  and  $z''_r$  are on the axis of  $M(g', id)$  and  $M(g'', id)$  respectively,  $M(g', id) \cap \partial B \times (t', t'') = \text{the } g'\text{-part of } M(g', id)$  and  $M(g'', id) \cap \partial B \times (t', t'') = \text{the } g''\text{-part of } M(g'', id)$ . Take a triangulation of  $C_r$  which underlies all the sets  $C'_r$  and  $C''_r$  and define the closed 4-cells covering  $C_r$  to be the sets  $C'_r$  and  $C''_r$  and the 4-simplexes which are not contained in  $C'_r$  and  $C''_r$  for all paths of  $\Gamma_i$  that cross  $C$ . We do that for all closed 4-cells  $C$  of  $Y_i$  and this way we define the closed 4-cells of the CW-structure of  $Y_i^r$ . As cells of  $\dim < 4$  we take the simplexes of  $\dim < 4$  of a triangulation underlying the closed 4-cells we already defined. It is easy to see that for every closed 4-cell  $C$  and every path  $\gamma \in \Gamma_i$  that crosses  $C$  there is a proper path  $\gamma_r^C$  in  $Y_{i+1}^r$  that begins at  $z'_r$ , ends at  $z''_r$  and crosses every closed 4-cell in  $C_r$ . Denote by  $\Gamma_{i+1}^r$  the collection of the paths  $\gamma_r^C$  for all closed 4-cells of  $Y_i$ . It is clear that taking  $\epsilon$  sufficiently small we may assume that the images of  $C_r$  under  $\omega_{i+1}^r$  are as small as we wish for all closed 4-cells of  $Y_i$ . Thus we may assume that the images of the cells of  $Y_{i+1}^r$  and the paths in  $\Gamma_{i+1}^r$  under  $\omega_{i+1}^r$  are as small as wish.

Now we will extend the CW-structure of  $Y_{i+1}^r$  to a CW-structure of  $Y_{i+1}$ , define a finite collection of proper paths  $\Gamma_{i+1}$  in  $Y_{i+1}$  and a finite closed cover  $\mathcal{F}_{i+1}$  of  $Y_{i+1}$ . Take a path  $\gamma \in \Gamma_i$  and let  $B \times [0, 1] \subset Y_{i+1}$  be the set used in the construction for the path  $\gamma$ . Then every path  $\gamma_r^C$  (for a closed 4-cell  $C$  of  $Y_i$  crossed by  $\gamma$ ) has two end points  $z'_r$  and  $z''_r$  in  $\partial B \times [0, 1]$  and these points are distinct for different cells  $C$  of  $Y_i$  crossed by  $\gamma$ . Denote by  $Z$  the collection of all the points  $z', z''$  for all the cells  $C$  crossed by  $\gamma$ . Take a PL-partition  $\mathcal{E}$  of  $B \times [0, 1] \cup \partial B \times [0, 1]$  into closed 3-cells such that each point in  $Z$  belongs to the interior of a cell in  $\mathcal{E}$ . Define the closed 4-cells covering  $B \times [0, 1]$  to be  $r_{i+1}^{-1}(E)$ ,  $E \in \mathcal{E}$ . Thus together with the closed 4-cells of  $Y_{i+1}^r$  we have defined the closed 4-cells of  $Y_{i+1}$  covering  $Y_{i+1}$  and as we already did before we define cells of  $\dim \leq 3$  to

be simplexes of  $\dim \leq 3$  of a triangulation of  $Y_{i+1}$  underlying the already defined closed 4-cells of  $Y_{i+1}$ . Extend every path  $\gamma_r^C \in \Gamma_{i+1}^r$  to a path  $\gamma_{i+1}^C$  by adding to the end points  $z'_r$  and  $z''_r$  of  $\gamma_r^C$  two paths going along  $r_{i+1}^{-1}(z'_r)$  and  $r_{i+1}^{-1}(z''_r)$  and connecting  $z'_r$  and  $z''_r$  with the points  $r_{i+1}^{-1}(z'_r) \cap B \times \{1\}$  and  $r_{i+1}^{-1}(z''_r) \cap B \times \{1\}$  respectively. It is clear that  $\gamma_{i+1}^C$  is a proper path of  $Y_{i+1}$ . For each cell  $E \in \mathcal{E}$  that does not contain a point of  $Z$  we will take any proper path  $\gamma_{i+1}^E$  contained in  $r_{i+1}^{-1}(E)$  and connects any two interior points of  $r_{i+1}^{-1}(E) \cap B \times \{1\}$ . Denote by  $\Gamma_{i+1}$  the collection of all the paths  $\gamma_{i+1}^E$  and  $\gamma_{i+1}^C$  we have constructed for all paths  $\gamma \in \Gamma_i$ . It is clear that every closed 4-cell of  $Y_{i+1}$  is crossed by a path in  $\Gamma_{i+1}$  and assuming that the partition  $\mathcal{E}$  of  $B \times [0, 1] \cup \partial B \times [0, 1]$  is fine enough we can also assume that the images of all the cells in  $Y_{i+1}$  and all the paths in  $\Gamma_{i+1}$  under  $\omega_i^{i+1}$  are as small as we wish. For each path in  $\Gamma_{i+1}$  consider the set which is the union of the closed 4-cells intersecting the path and define the cover  $\mathcal{F}_{i+1}$  to be the collection of such sets for all the paths in  $\Gamma_{i+1}$ . Thus we have constructed  $\Gamma_{i+1}$  satisfying (iv) and we may assume that  $\mathcal{F}_{i+1}$  satisfies (v). The construction is completed.

The last step of the proof is to verify that  $\dim_{\mathbb{Q}} Y \leq 3$ . Note that for closed 4-cells  $C$  of  $Y_i$ , the sets  $q_i(C \cap Y'_i) \setminus q_i(\partial C \cap Y'_i)$  are open in  $Y_{i+1}^r$ , disjoint for different cells  $C$  and the complement in  $Y_{i+1}^r$  of the union of these sets is a PL-subcomplex of  $\dim \leq 3$ . Recall that we may assume that the images of  $q_i(C \cap Y'_i)$  under  $\omega_{i+1}^r$  are as small as we wish. Then we may assume that for the map  $r_{i+1} \circ \omega_{i+1} : Y \rightarrow Y_{i+1}^r$  the fibers of  $r_{i+1} \circ \omega_{i+1}$  and the preimages of the sets  $q_i(C \cap Y'_i)$  for closed 4-cells  $C$  of  $Y_i$  are as small as we wish if  $i$  is sufficiently large. This together with (\*) implies that  $H^4(Y, Y'; \mathbb{Q}) = 0$  for every closed subset  $Y'$  of  $Y$  and hence  $\dim_{\mathbb{Q}} Y \leq 3$ . The proof is completed. ■

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